PRESSURE OF AN AXIALLY SYMMETRIC CIRCULAR DIE ON AN ELASTIC HALF-SPACE

(DAVLENIE OSESIMMETRICHNOGO KOL'TSEVOGO SHTAMPA NA UPRUGOE POLUPROSTRANSTVO)

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Considered is the problem of the pressure of a rigid die, having a plan form of a circular concentric ring, upon an elastic half-space.

This problem attracted the attention of scientific workers. For example, the recent work of Egorov [1] and Aleksandrov [2] may be mentioned. The difficulties known to occur in such type of problems require the establishment of an effective [approximate] solution. In one of the earlier works of Lebedev [3], special functions are introduced for this purpose, with the aid of which the solution of several boundary value problems for the annular region becomes possible.

The integral transforms used in the present work, with the aid of which Mossakovskii [4] reduces the axially symmetric problem to the problem of linear conjugation in the plane of the complex variable, were employed for the derivation of a Fredholm-type integral equation with respect to the boundary value (in the region of contact) of one of the unknown functions; if this is known it is not difficult to find the pressure by means of a quadrature. For the Fredholm equation there is constructed an approximate solution with the aid of known, for this case approximate, methods.

The surface of the die after impression is considered to be axially symmetric (the equation of this surface is given), the friction in the region of contact is not taken into account, pressure outside of the die is absent. The law of pressure distribution underneath the die is found.*

* In article [5] Rostovtsev indicated an error made by Gubenko in his paper [6]. The formulas for the pressure underneath a circular die with a plane base, obtained by means of formal application of fractional differentiation, are erroneous.

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1. As is known, the solution of the problem [7, 8] reduces to finding the normal derivative $F_z'(\rho, 0)$ in the region of contact of a certain function $F(\rho, z)$, harmonic in the elastic half-space, vanishing at infinity and satisfying the following conditions at the boundary of the elastic half-space:

$$F_z'(\rho, 0) = 0, \quad 0 < \rho < b, \quad a < \rho < \infty$$
$$F(\rho, 0) = f(\rho), \quad b < \rho < a \tag{1.1}$$

where a and b are the outer and the inner radii of the ring, respectively, ρ is the polar radius, $f = f(\rho)$ is the equation of the die surface.

The origin of the coordinates is taken here at the center of the ring, the z-axis is directed into the half-space.

The pressure $p(\rho)$ underneath the die is determined by the formula

$$p(\rho) = \frac{E}{2(1-\nu^2)} F_z'(\rho, 0) \qquad (b < \rho < a) \tag{1.2}$$

where E is the modulus of elasticity and ν is Poisson's ratio.

2. We make use of the following formulas [3]:

$$u_{ix}'(x, z) = 4\frac{\partial}{\partial x}\int_0^x \frac{Fi(\rho, z)}{\sqrt{x^2-\rho^2}}\rho d\rho \qquad (i = 1, 2)$$
$$u_{zi}'(x, z) = 4\frac{\partial}{\partial x}\int_x^\infty \frac{Fi(\rho, z)}{\sqrt{\rho^2-x^2}}\rho d\rho \tag{2.1}$$

and the inverse

$$Fi(\rho, z) = \frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int_0^\rho \frac{u_i(x, z)}{\sqrt{\rho^2-x^2}} x dx$$
$$F_{iz}'(\rho, z) = -\frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int_\rho^\infty \frac{u_{ix}'(x, z)}{\sqrt{x^2-\rho^2}} x dx \tag{2.2}$$

where $u_i(x, z)$ are harmonic functions in the plane (x, z) and antisymmetric with respect to x; $Fi(\rho, z)$ are harmonic functions in half-space and symmetric with respect to ρ.

3. In the general case it should be assumed that the function $f(\rho)$ may be represented in the form

$$f(\rho) = \ldots + a_2\rho^2 + a_1\rho + a_0 + \frac{a_{-1}}{\rho} + \frac{a_{-2}}{\rho^2} + \frac{a_{-3}}{\rho^3} + \ldots$$

We put

$$f_1(\rho) = a_0 + a_1\rho + a_2\rho^2 + \ldots, \qquad f_2(\rho) = \frac{a_{-1}}{\rho} + \frac{a_{-2}}{\rho^2} + \frac{a_{-3}}{\rho^3} + \ldots$$

Then

$$f(\rho) = f_1(\rho) + f_2(\rho)$$

Such a representation of the function $f(\rho)$ is obviously unique and the function $f_1(\rho)$ may be extended to zero and the function $f_2(\rho)$ to infinity.

Let us introduce now two functions $F_1(\rho, z)$ and $F_2(\rho, z)$ which are harmonic in the elastic half-space, such that

$$F_1(\rho, z) + F_2(\rho, z) = F(\rho, z)$$

$$F_1(\rho, 0) = f_1(\rho) \quad (0 < \rho < a), \qquad F_2(\rho, 0) = f_2(\rho) \quad (b < \rho < \infty)$$

Then the boundary conditions (1.1) attain the form

$$F_{1z}'(\rho, 0) + F_{2z}'(\rho, 0) = 0 \qquad (0 < \rho < b,\ a < \rho < \infty) \tag{3.1}$$

$$F_1(\rho, 0) = f_1(\rho) \quad (0 < \rho < a), \qquad F_2(\rho, 0) = f_2(\rho) \quad (b < \rho < \infty)$$

By Formulas (2.1) the last conditions are transformed for functions $u_i(x, z)$ as follows:

$$\begin{aligned} &u_{1x}'(x, 0) + u_{2x}'(x, 0) = 0, \quad a < |x|; \qquad && u_{1x}'(x, 0) = g_1(x), \quad |x| < a \\ &u_{1z}'(x, 0) + u_{2z}'(x, 0) = 0, \quad |x| < b; \qquad && u_{2z}'(x, 0) = g_2(x), \quad |b| < |x| \end{aligned} \tag{3.2}$$

where

$$g_1(x) = 4\frac{\partial}{\partial x}\int_0^x \frac{f_1(\rho)}{\sqrt{x^2 - \rho^2}}\rho d\rho, \qquad g_2(x) = 4\frac{\partial}{\partial x}\int_x^\infty \frac{f_2(\rho)}{\sqrt{\rho^2 - x^2}}\rho d\rho \tag{3.3}$$

4. In finding $u_i(x, y)$ $(y = z)$ from the boundary conditions one can proceed in different ways. Let us use the following procedure. Let us assume that two functions $Q_1(x, y, t)$ and $Q_2(x, y, t)$ are found which are harmonic in the half-plane xy, antisymmetric with respect to x, vanishing at infinity and satisfying along the straight line $y = 0$ the following conditions:

$$Q_{1x}'(x, 0, t) = 1, \qquad |x| < t, \qquad Q_{1x}'(x, 0, t) = 0, \qquad |x| > t$$

$$Q_{2y}'(x, 0, t) = 1, \qquad |x| < t; \qquad Q_{2y}'(x, 0, t) = 0, \qquad |x| > t$$

where t is a parameter.

Then the derivatives $u_{1x}'(x, y)$ and $u_{2y}'(x, y)$ may be represented in the form

$$\begin{aligned} u_{1x}'(x, y) = &-\int_0^a g_1'(t) Q_{1x}'(x, y, t)\, dt + \int_a^\infty u_{2xt}''(t, 0) Q_{1x}'(x, y, t)\, dt + \\ &+ [g_1(a) + u_{2x}'(a, 0)] Q_{1x}'(x, y, a) \\ u_{2y}'(x, y) = &\int_0^b u_{1yt}''(t, 0) Q_{2y}'(x, y, t)\, dt - \int_a^\infty g_2'(t) Q_{2y}'(x, y, t)\, dt - \\ &- [g_2(b) + u_{1y}'(b, 0)] Q_{2y}'(x, y, b) \end{aligned} \qquad (4.1)$$

The functions $Q_1(x, y, t)$ and $Q_2(x, y, t)$ are easily determined. They are of the following form:

$$\begin{aligned} Q_{1x}'(x, y, t) + iQ_{1y}'(x, y, t) &= -\frac{i}{\pi} \ln \frac{z-t}{z+t} \\ Q_{2x}'(x, y, t) + iQ_{2y}'(x, y, t) &= \frac{2}{\pi} \ln \frac{\sqrt{z^2 - t^2}}{z} \end{aligned} \qquad (z = x + iy) \qquad (4.2)$$

Equations (4.1) may be represented in the form

$$\begin{aligned} u_{1y}'(x, y) = &-\int_0^a g_1'(t) Q_{1y}'(x, y, t)\, dt + \int_a^\infty u_{2xt}''(t, 0) Q_{1y}'(x, y, t)\, dt + \\ &+ [g_1(a) + u_{2x}'(a, 0)] Q_{1y}'(x, y, a) \\ u_{2x}'(x, y) = &\int_0^b u_{1yt}''(t, 0) Q_{2x}'(x, y, t)\, dt + \int_b^\infty g_2'(t) Q_{2x}'(x, y, t)\, dt - \\ &- [g_2(b) + u_{1y}'(b, 0)] Q_{2x}'(x, y, b) \end{aligned} \qquad (4.3)$$

Integrating (4.3) by parts and taking (4.2) into account we obtain

$$\begin{aligned} u_{1y}'(x, y) &= \frac{2}{\pi} \int_0^a g_1(t) \operatorname{Im} \left\{\frac{iz}{z^2 - t^2}\right\} dt - \frac{2}{\pi} \int_a^\infty u_{2x}'(t, 0) \operatorname{Im} \left\{\frac{iz}{z^2 - t^2}\right\} dt \\ u_{2x}'(x, y) &= \frac{2}{\pi} \int_0^b u_{1y}'(t, 0) \operatorname{Re} \left\{\frac{t}{z^2 - t^2}\right\} dt - \frac{2}{\pi} \int_b^\infty g_2(t) \operatorname{Re} \left\{\frac{t}{z^2 - t^2}\right\} dt \end{aligned} \qquad (4.4)$$

For $y = 0$ the last equations take on the following form:

$$u'_{1y}(x, 0) = \frac{2x}{\pi}\int_0^a \frac{g_1(t)}{x^2 - t^2}\,dt - \frac{2x}{\pi}\int_a^\infty \frac{u'_{2x}(t, 0)}{x^2 - t^2}\,dt$$
$$u'_{2x}(x, 0) = \frac{2}{\pi}\int_0^b \frac{u'_{1y}(t, 0)}{x^2 - t^2}\,t\,dt - \frac{2}{\pi}\int_b^\infty \frac{g_2(t)}{x^2 - t^2}\,t\,dt \tag{4.5}$$

valid for arbitrary x.

5. The derivative $F_z'(\rho, 0)$ may be determined, for example, from the first formula (2.2). Taking (3.2) into account, we obtain

$$F_z'(\rho, 0) = \frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int_b^\rho \frac{g_2(x) + u'_{1z}(x, 0)}{\sqrt{\rho^2 - x^2}}\,x\,dx \tag{5.1}$$

We substitute into (5.1) the value of $u'_{1y}(x, 0)$ from (4.5). In the double integral obtained we change the order of integration, assuming $b < \rho < a$. Taking then (1.2) into account we find

$$\begin{aligned} p(\rho) = \frac{E}{2(1 - \nu^2)\pi^2}\Bigg\{&\frac{bA}{\rho^2\sqrt{\rho^2 - b^2}} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{\pi}{2}\int_b^\rho \frac{g_2(x)}{\sqrt{\rho^2 - x^2}}\,x\,dx - \right. \\ &- \int_\rho^a \frac{t}{\sqrt{t^2 - \rho^2}}\cot^{-1}\frac{b}{t}\frac{\sqrt{t^2 - \rho^2}}{\sqrt{\rho^2 - b^2}}\,g_1(t)\,dt + \\ &+ \frac{1}{2}\int_0^\rho \frac{t}{\sqrt{\rho^2 - t^2}}\ln\left|\frac{t\sqrt{\rho^2 - b^2} + b\sqrt{\rho^2 - t^2}}{t\sqrt{\rho^2 - b^2} - b\sqrt{\rho^2 - t^2}}\right]g_1(t)\,dt + \\ &\left. + \int_a^\infty \frac{t}{\sqrt{t^2 - \rho^2}}\cot^{-1}\frac{b}{t}\frac{\sqrt{t^2 - \rho^2}}{\sqrt{\rho^2 - b^2}}\,u'_{2x}(t, 0)\,dt\right\} \end{aligned} \tag{5.2}$$

where

$$A = \int_0^a g_1(t)\,dt - \int_a^\infty u'_{2x}(t, 0)\,dt$$

We note that for $b = 0$ Formula (5.2) takes on the simple form

$$p(\rho) = -\frac{E}{2(1 - \nu^2)}\frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int_\rho^a \frac{g_1(t)}{\sqrt{t^2 - \rho^2}}\,t\,dt$$

which coincides with the solution of the problem of a circular die, if (3.3) is taken into account.

6. We proceed now to find the derivative $u'_{2x}(x, 0)$ in the region (a, ∞).

We eliminate the derivative $u'_{1y}(x, 0)$ from (4.5) and obtain then the following integral equation for $u'_{2x}(x, 0)$:

$$u'_{2x}(x, 0) = \frac{2}{\pi^2}\int_a^\infty \frac{1}{x^2 - s^2}\left(s \ln\frac{s+b}{s-b} - x\ln\frac{x+b}{x-b}\right) u'_{2x}(s, 0)\, ds + \Phi(x) \quad (6.1)$$

where

$$\Phi(x) = -\frac{2}{\pi^2}\int_0^a \frac{1}{x^2 - s^2}\left(s \ln\frac{s+b}{s-b} - x\ln\frac{x+b}{x-b}\right) g_1(s)\, ds - \frac{2}{\pi}\int_b^\infty \frac{g_2(t)}{x^2 - t^2}\, dt \quad (6.2)$$

In solving Equation (6.1) we use the approximation

$$\ln\frac{a+b}{a-b} \approx 2\left(\frac{b}{a} + \frac{1}{3}\frac{b^3}{a^3} + \ldots + \frac{1}{2N+1}\frac{b^{2N+1}}{a^{2N+1}}\right) \qquad (b < a) \qquad (6.3)$$

The value of N is selected in such a manner as to insure a certain prescribed accuracy. Then with an accuracy which is not smaller than $x < a$, $s < a$ we have

$$\begin{aligned} \ln\frac{s+b}{s-b} &\approx 2\left(\frac{b}{s} + \frac{1}{3}\frac{b^3}{s^3} + \ldots + \frac{1}{2N+1}\frac{b^{2N+1}}{s^{2N+1}}\right) \\ \ln\frac{x+b}{x-b} &\approx 2\left(\frac{b}{x} + \frac{1}{3}\frac{b^3}{x^3} + \ldots + \frac{1}{2N+1}\frac{b^{2N+1}}{s^{2N+1}}\right) \end{aligned} \qquad (6.4)$$

Taking (6.4) into account, the kernel of the integral equation (6.1) may be represented in the form

$$\frac{1}{x^2 - s^2}\left(s \ln\frac{s+b}{s-b} - x\ln\frac{x+b}{x-b}\right) \approx \frac{C_2(s)}{x^2} + \frac{C_4(s)}{x^4} + \ldots + \frac{C_{2N}(s)}{x^{2N}} \qquad (6.5)$$

where

$$C_{2k}(s) = 2b^{2k-1}\left(\frac{1}{2k+1}\frac{b^2}{s^2} + \frac{1}{2k+3}\frac{b^4}{s^4} + \ldots + \frac{1}{2N+1}\frac{b^{2(N-k+1)}}{s^{2(N-k+1)}}\right)_{(k=1,\ldots,N)} \quad (6.6)$$

Substituting (6.5) into (6.1), we obtain the following representation:

$$u'_{2x}(x, 0) \approx \frac{2}{\pi^2}\left[\frac{z_2}{x^2} + \frac{z_4}{x^4} + \ldots + \frac{z_{2N}}{x^{2N}}\right] + \Phi(x) \qquad (6.7)$$

where

$$z_{2k} = \int_a^\infty C_{2k}(s)\, u'_{2x}(s, 0)\, ds \qquad (k = 1, \ldots, N) \qquad (6.8)$$

Let us substitute (6.7) into (6.8). Then for z_{2k} we obtain the system of equations

$$\begin{aligned} z_2 &= A_{22} z_2 + A_{42} z_4 + \ldots + A_{2n2} z_{2n} + \ldots + A_{2N2} z_{2N} + B_2 \\ z_4 &= A_{24} z_2 + A_{44} z_4 + \ldots + A_{2n4} z_{2n} + \ldots + A_{2N4} z_{2N} + B_4 \end{aligned}$$

$$\begin{aligned} &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_{2k} &= A_{22k}\, z_2 + A_{42k}\, z_4 + \ldots + A_{2n2k}\, z_{2n} + \ldots + A_{2N2k} z_{2N} + B_{2k} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_{2N} &= A_{22N}\, z_2 + A_{42N}\, z_4 + \ldots + A_{2n2N} z_{2n} + \ldots + A_{2N2N}\, z_{2N} + B_{2N} \end{aligned} \tag{6.9}$$

where

$$A_{2n\,2k} = \frac{2}{\pi^2}\int\limits_a^\infty \frac{C_{2k}(s)}{s^{2n}}\,ds, \qquad B_{2k} = \int\limits_a^\infty C_{2k}(s)\,\Phi(s)\,ds \tag{6.10}$$

7. Let us consider an example. Let $f(\rho) = h$ const (die with a plane base), and

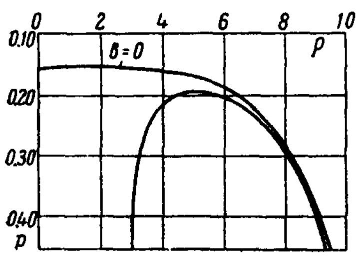
$$\frac{b}{a} \leqslant \frac{e-1}{e+1} = 0.462 \qquad (e=2.7128\ldots) \tag{7.1}$$

In this case we must assume $f_1(\rho) = h$, $f_2(\rho) = 0$. From (3.3) we find $g_1(x) = 4h$, $g_2(x) = 0$. By (6.3) and (7.1) we find

$$\ln\frac{a+b}{a-b} \approx 2\left(\frac{b}{a} + \frac{1}{3}\frac{b^3}{a^3}\right), \qquad \text{or} \qquad N = 1 \tag{7.2}$$

The error is here not larger than 1%. The system (6.9) takes on the form:

$$z_2 = A_{22}\, z_2 + B_2, \qquad \text{or} \qquad z_2 = \frac{B_2}{1 - A_{22}} \tag{7.3}$$



By Formulas (6.6) and (6.10) we find

$$C_2(s) = \frac{2}{3}\frac{b^3}{s^2}, \quad A_{22} = \frac{2}{\pi^2}\int\limits_a^\infty \frac{C_2(s)}{s^2}\,ds = \frac{4}{9\pi^2}\left(\frac{b}{a}\right)^3 \tag{7.4}$$

From (6.2) we find

$$\Phi(s) = \frac{8h}{\pi^2}\int\limits_0^b \ln\frac{a+t}{a-t}\,\frac{t\,dt}{s^2-t^2} \tag{7.5}$$

Taking into account (7.4), (7.5) and (7.2) we find from (6.11)

$$B_2 = \int\limits_a^\infty C_2(s)\,\Phi(s)\,ds = \frac{32h}{27\pi^2}\left(\frac{b}{a}\right)^4\left(1 + \frac{1}{5}\frac{b^2}{a^2}\right)b^2 \tag{7.6}$$

On the basis of (7.3), (7.4) and (7.6) we find

$$z_2 = \frac{32h}{27\pi^2}\left(\frac{b}{a}\right)^4 \frac{1 + \dfrac{1}{5}\left(\dfrac{b}{a}\right)^3}{1 - \dfrac{4}{9\pi^2}\left(\dfrac{b}{a}\right)^3}\, b^2 \tag{7.7}$$

From (6.7), (7.5) and (7.7) we obtain

$$u_{2x}'(x,0)=\frac{64h}{27\pi^4}\left(\frac{b}{a}\right)^4\frac{1-\frac{1}{5}\left(\frac{b}{a}\right)^2}{1-\frac{4}{9\pi^2}\left(\frac{b}{a}\right)^3}\left(\frac{b}{x}\right)^2+\frac{8h}{\pi^2}\int_0^b\ln\frac{a+t}{a-t}\frac{tdt}{x-t^2}$$

or taking (7.2) into account we find

$$u_{2x}'(x,0)=c\left(\frac{b}{x}\right)^2,\qquad c=\frac{16h}{3\pi^2}\frac{b}{a}\frac{1+\frac{4}{45\pi^2}\left(\frac{b}{a}\right)^5}{1-\frac{4}{9\pi^2}\left(\frac{b}{a}\right)^3}\tag{7.8}$$

In the expression for c the third factor is small as compared to unity; therefore

$$u_{2x}'(x,0)\approx\frac{16h}{3\pi^2}\left(\frac{b}{a}\right)\left(\frac{b}{x}\right)^2\tag{7.9}$$

Formula (5.2) for $g_1(x) = 4h$ and $g_2(x) = 0$ takes on the form

$$p(\rho)=\frac{E}{2(1-\nu^2)\pi^2}\left[\frac{4h}{\sqrt{a^2-\rho^2}}\cot^{-1}\frac{b}{a}\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}}-\frac{B(\rho)}{\sqrt{\rho^2-b^2}}+\right.$$
$$\left.+\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_a^\infty\frac{t}{\sqrt{t^2-\rho^2}}\cot^{-1}\frac{b}{t}\frac{\sqrt{t^2-\rho^2}}{\sqrt{\rho^2-b^2}}u_{2x}'(t,0)\,dt\right]\tag{7.10}$$

where

$$B=2h\ln\frac{a-b}{a+b}+\frac{b}{\rho^2}\int_a^\infty u_{2x}'(t,0)\,dt\tag{7.11}$$

Let us substitute (7.11) and (7.9) into (7.10). The integral, entering into (7.10), is integrated subsequently, first by parts and then taking expansion (7.2) into account. We obtain

$$p(\rho)=\frac{4hE}{2(1-\nu^2)\pi^2}\left[\frac{1}{\sqrt{a^2-\rho^2}}\cot^{-1}\frac{b}{a}\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}}-\frac{B_1(\rho)}{\sqrt{\rho^2-b^2}}-\frac{4b^3}{3\pi^2a\rho}\frac{dJ}{d\rho}\right]\tag{7.12}$$

where

$$B_1(\rho)=-\frac{1}{2}\ln\frac{a+b}{a-b}+\frac{4}{3\pi^2}\left(\frac{b}{a}\right)^2\left(\frac{b}{\rho}\right)^2\tag{7.13}$$

$$J=\frac{1}{\rho}\cot^{-1}\frac{\sqrt{a^2-\rho^2}}{\rho}\cot^{-1}\frac{b}{a}\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}}-\tag{7.14}$$
$$-\frac{b\sqrt{\rho^2-b^2}}{a^3\rho^4}\left[\left(a^2\rho^2+\frac{2}{3}b^2a^2+\frac{1}{9}b^2\rho^2\right)\left(\rho-\sqrt{a^2-\rho^2}\right)-\frac{2}{9}b^2\rho^2\sqrt{a^2-\rho^2}\right]$$

In particular, for $b = 0$ Formula (7.12) takes on the simple form

$$p(\rho) = \frac{Eh}{\pi(1-\nu^2)} \frac{1}{\sqrt{a^2-\rho^2}}$$

Let us remark in conclusion that if the die is so wide that the approximation

$$\ln\frac{a+b}{a-b} \approx 2\frac{b}{a}$$

is valid then, as is easily verified, the derivative $u'_{2x}(x, 0) = 0$. Then Formula (7.12) takes on the form

$$p(\rho) = \frac{4hE}{2(1-\nu^2)\pi^2}\left[\frac{1}{\sqrt{a^2-\rho^2}} \cot^{-1}\frac{b}{a}\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}} + \frac{b}{a\sqrt{\rho^2-b^2}}\right]$$

The graph of pressure distribution underneath the annular die for $b/a = 0.3(a = 10)$, constructed by Formula (7.12) with an accuracy within the factor $2hE/(1-\nu^2)\pi^2$, is shown in the figure.

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